System Identification Under Bounded Noise: Optimal Rates Beyond Least Squares

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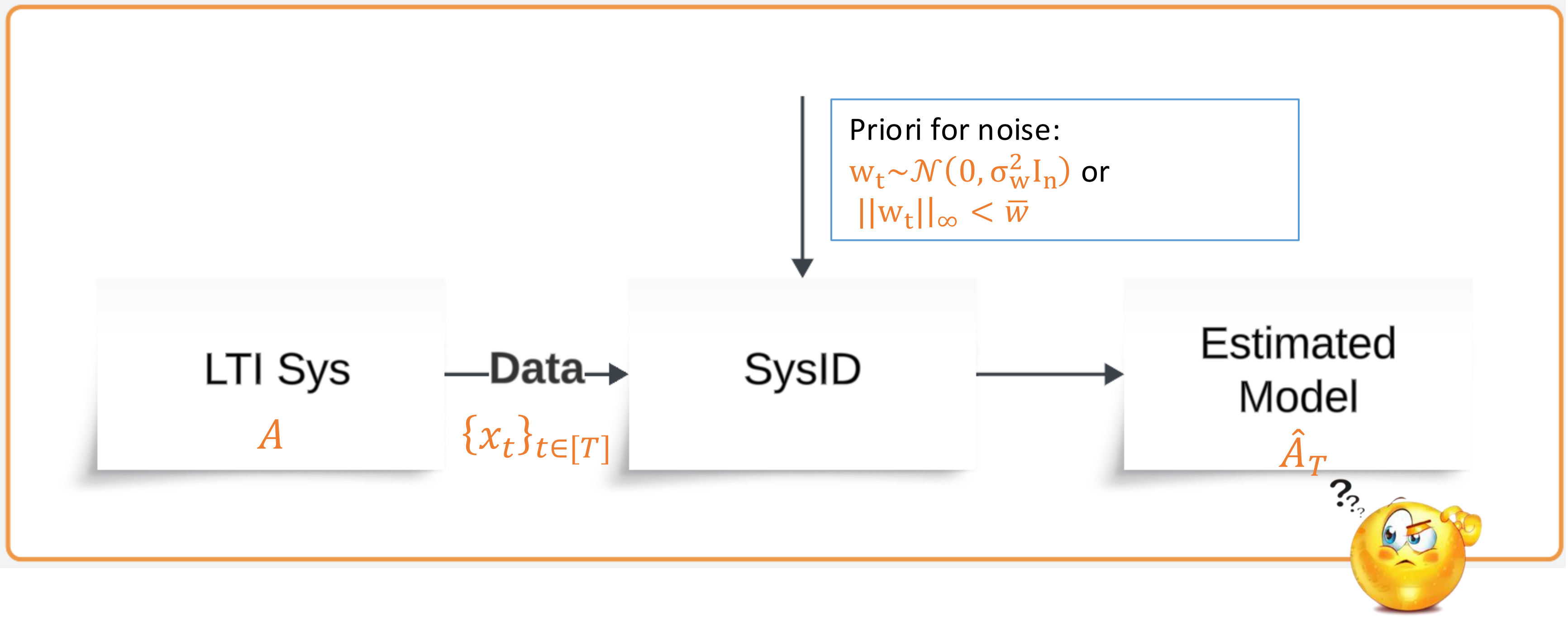
System identification is important for physical intelligence

ANYmal from ETH
LeCAR lab from CMU

Consider a linear time-invariant (LTI) system:

$$x_{t+1} = Ax_t + w_t,$$

where A is unknown, $x_t, w_t \in \mathbb{R}^n$.



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where A is unknown, $x_t, w_t \in \mathbb{R}^n$.

Theorem 1 (Simchowitz et al., COLT 2018). When the noise w_t is Gaussian, the ordinary least squares (OLS) estimator is minimax optimal.

In safety-critical scenarios, bounded noise is more common.

Two questions:

1. Is OLS still minimax optimal for bounded noise?

2. Which estimator is minimax optimal for bounded noise?

Consider an LTI system:

$$\mathrm{x}_{\mathrm{t}+1} = \mathrm{A}\mathrm{x}_{\mathrm{t}} + \mathrm{w}_{\mathrm{t}},$$

where A is unknown, $\mathrm{x}_{\mathrm{t}}, w_t \in \mathbb{R}^{\mathrm{n}}$. Assume:

- **(bounded noise)** $||\mathrm{w}_{\mathrm{t}}||_{\infty} < \bar{w}\ and\ \mathrm{w}_{\mathrm{t}}$ are i.i.d. ∀t,
- (**stable**) $\rho(A) < 1$, where ρ(·) := the spectral radius
- (**boundary probability**) $\forall \epsilon \in [0, \bar{w}]$, $\exists C > 0$, such that $\forall j \in [n]$,

$$\max\left(P\left(w_t^{(j)} < -\bar{w} + \epsilon\right), P\left(w_t^{(j)} > \bar{w} - \epsilon\right)\right) < C\epsilon.$$

Theorem 2 (**our work**). Given a single trajectory $\{x_t\}_{t\in[T]}$. $\mathcal{F}_T$:= the σ-algebra generated by $\{x_t\}_{t\in[T]}\ and\ \hat{A}_T$:= a $\mathcal{F}_T$ -measurable estimator of A. Then, $\forall \delta \in (0,1)$ and ∀ small $\epsilon > 0$,

$$\sup_{\hat{A}_T} \inf_{A} P(||\hat{A}_T - A||_2 < \epsilon) \geq 1 - \delta \quad \text{only if} \quad T > \frac{1}{4\bar{w}C\epsilon}\left(1 - \frac{2\delta}{n}\right).$$

(The best estimator can achieve $\Omega(\frac{1}{\epsilon})$)

The ordinary least squares (OLS) for scalar case:

$$\hat{a}_T^{OLS} = \mathrm{argmin}_{\mathrm{a}} \sum_{t=1}^{T-1} \left|\left|x_{t+1} - a x_t\right|\right|^2$$

Theorem 3 (our work). Assume $|a| < 1$. Then, $\forall \delta \in (0,1)$ and small $\epsilon > 0$,

$$P\left(\left|\left|\hat{a}_T^{OLS} - a\right|\right|_2 < \epsilon\right) \geq 1 - \delta, \quad \text{only if } \; T > \Omega\left(\frac{1}{\epsilon^2}\right).$$

($\Omega\left(\frac{1}{\epsilon^2}\right) > \Omega\left(\frac{1}{\epsilon}\right)$, OLS is not optimal for bounded noise)

The set membership estimator (SME) based on $\{x_t\}_{t\in[T]}$

$$\mathcal{S}_{\mathrm{T}} = \left\{A \in \mathbb{R}^{n\times n}: \left|\left|x_{t+1} - A x_t\right|\right|_\infty \leq \bar{w}, \forall t \in [T-1]\right\}.$$

Theorem 4 (Li & Yu et al., ICML 2024). $\forall \delta \in (0,1)$ and small $\epsilon > 0$,

$$\forall \hat{A}_T \in \mathcal{S}_{\mathrm{T}}, \; P\left(\left|\left|\hat{A}_T - A\right|\right|_2 < \epsilon\right) \geq 1 - \delta, \quad \text{if} \quad T > \Omega\left(\frac{1}{\epsilon}\right).$$

(SME achieves the optimal $\Omega\left(\frac{1}{\epsilon}\right)$)

Basic Intuition

For two length-T trajectories of two different systems that differ by ϵ,

- **Gaussian** → KL divergence = $O(T\epsilon^2)$;
- **Bounded** → Total variation (TV) distance = $O(T\epsilon)$.

(ChatGPT can't handle it ☺)

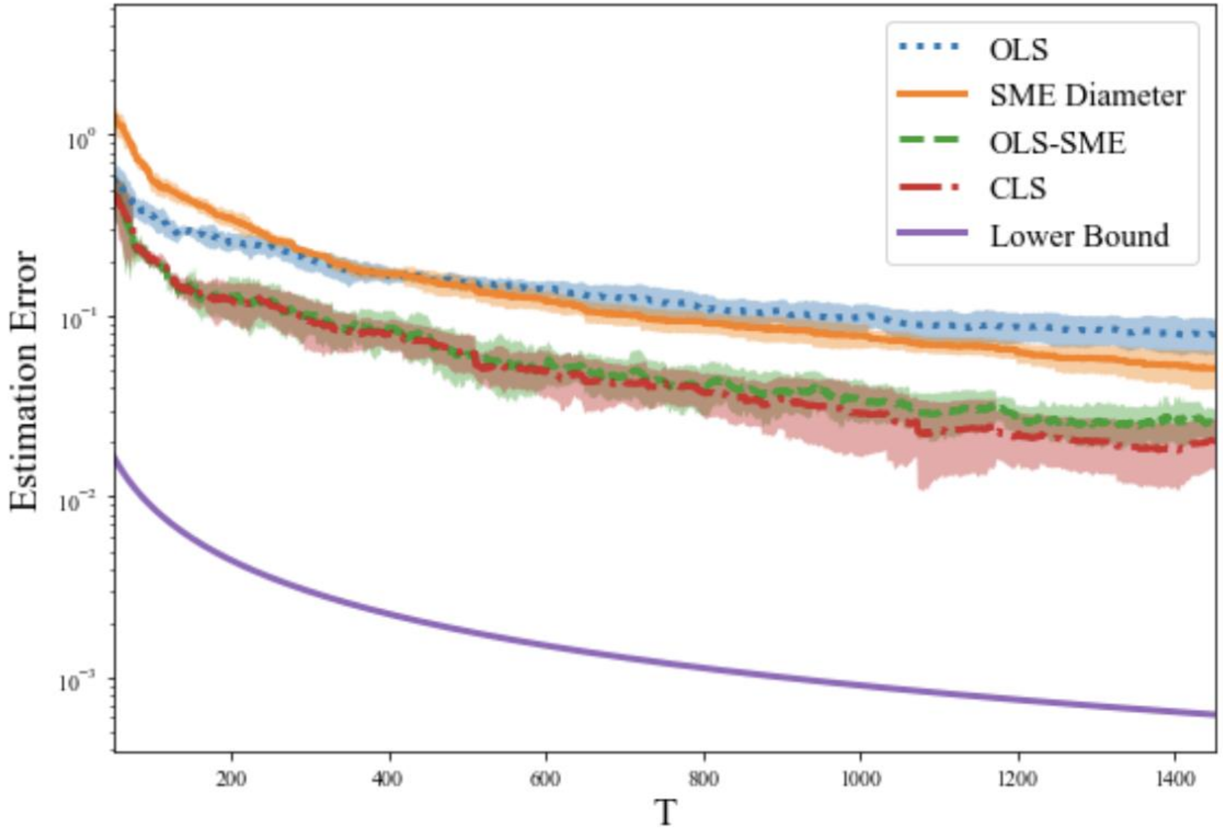
Simulation

OLS: $\min_{A} \sum_{t=1}^{T-1} ||x_{t+1} - Ax_t||^2$

OLS-SME: $\min_{A \in \mathcal{S}_T} ||A - \hat{A}_T^{OLS}||$

Constrained Least Square (CLS):

$$\min_{A \in \mathcal{S}_T} \sum_{t=1}^{T-1} ||x_{t+1} - Ax_t||^2$$



OLS achieves $O(\frac{1}{\sqrt{T}})$ ☹

SME, OLS-SME, and CLS achieve $O(\frac{1}{T})$ ☺

CLS > OLS-SME > SME > OLS

Summary and Future Work:

		Minimax Lower Bound	Lower Bound for OLS
Regression	Gaussian	$\Omega(1/\sqrt{T})$ (Wainwright, 2019)	$\Omega(1/\sqrt{T})$ (Mourtada, 2022)
	Bounded	$\Omega(1/T)$ (Yi & Neykov, 2024)	$\Omega(1/\sqrt{T})$ (Rudelson & Vershynin, 2008)
LTI Sys Id	Gaussian	$\Omega(1/\sqrt{T})$ (Jedra & Proutiere, 2019)	$\Omega(1/\sqrt{T})$ (Tu et al., 2024)
	Bounded	$\Omega(1/T)$ (**Thm. 2**)	$\Omega(1/\sqrt{T})$ (**Thm. 3**)

1. The state dimension n and the failure probability δ dependencies of our minimax lower bound are loose.

2. In experiments, CLS > SME. The tight sample complexity upper bound of CLS is unclear.

(ChatGPT might handle them ☺)

Thank you!

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